$$\boxed{\text{Use: } B \in (\mathrm{CAlg}^{\mathrm{aug}}_{k})^{\mathrm{sm}}, \; \text{~~As~~} \cdot \Rightarrow B \longrightarrow C^{\vee}\mathfrak{D}(B) \text{ is equiv.}}$$

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Def'n good Lie algebra $\mathfrak{g}$: • cofibrant, $\exists V_{\bullet} \subseteq \mathfrak{g}_{\bullet}$ st

1. $\forall n \in \mathbb{Z}$, V_n is f.dim'l
2. $V_n = 0$ $n \geq 0$
3. $\mathfrak{g}_{\bullet}$ is a free Lie algebra

Thm $\mathrm{Lie}_k^{\mathrm{good}} \subseteq \mathrm{Lie}_k$ full sf cat spanned by good Lie algebras:

(a) $\forall \mathfrak{g} \in \mathrm{Lie}^{\mathrm{good}}$, $\mathfrak{g} \longrightarrow \mathfrak{D}C^{\vee}\mathfrak{g}$ is equiv

(from Prop. last week)

(b) $\forall n \geq 1$ $\exists$ $\mathfrak{g}^{(n)} \in \mathrm{Lie}^{\mathrm{good}}$, and an equivalence in $\mathrm{CAlg}^{\mathrm{aug}}$
$k \oplus k[n] \simeq C^{\vee}\mathfrak{g}^{(n)}$

$$\boxed{\mathfrak{g}^{(n)} := \mathrm{Free}_{\mathrm{Lie}}(k[-n-1])}$$

(c) V_n: $\mathfrak{g}^{(n)} \simeq \mathfrak{D}(k \oplus k[n]) \longrightarrow \mathfrak{D}(k) \simeq 0$

$$\begin{tikzcd} \mathfrak{g}^{(n)} \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathfrak{g} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathfrak{g}' \end{tikzcd}$$

$\mathfrak{g} \in \mathrm{Lie}^{\mathrm{good}} \Rightarrow \mathfrak{g}' \in \mathrm{Lie}^{\mathrm{good}}$

(Recall:
$$\begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & k \arrow[d] \\ A \arrow[r] & k \oplus k[n] \end{tikzcd}$$
A small $\Rightarrow$ B small

Prop Properties of $\mathfrak{D}$:

1. $\mathfrak{D}(k) \simeq 0$
2. $A \in \mathrm{CAlg}^{aug}$, $A = C^*(\mathfrak{g})$, $\mathfrak{g} \in \mathrm{Lie}^{good}$ $\Rightarrow$ $A \to C^*\mathfrak{D}(A)$ is an equiv in CAlg^{aug}
3. $A \in \mathrm{CAlg}^{aug}$ is small $\to$ $\mathfrak{D}(A) \in \mathrm{Lie}^{good}$, $A \to C^*\mathfrak{D}(A)$ is equiv
4. σ

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d, "\phi"] \\
A \arrow[r] & B
\end{tikzcd}$$

is pb in CAlg^{aug}, A, B, ϕ small $\Rightarrow$ $\mathfrak{D}(\sigma)$ is po.

Pf 1. $C^*(0) \simeq k$, 0 is good Lie alg $\xRightarrow{\text{Thm (a)}}$ $0 \simeq \mathfrak{D}(C^*(0)) = \mathfrak{D}(k)$

2. $\mathfrak{g} \in \mathrm{Lie}^{good}$ $\xRightarrow{\text{Thm (a)}}$ $\mathfrak{g} \xrightarrow{\simeq} \mathfrak{D}C^*(\mathfrak{g})$

$\Rightarrow A = C^*(\mathfrak{g}) \xleftarrow{\simeq} C^*\mathfrak{D}C^*(\mathfrak{g}) = C^*\mathfrak{D}A$

is left homotopy inverse of $\underset{=}{u}: A \longrightarrow C^*\mathfrak{D}A$
b/c u is unit of adj

$\Rightarrow$ u is equivalence as well.

3. A small cdga $\Rightarrow$ can choose a sequence of elementary morphisms $A = A^{(0)} \to A^{(1)} \to \cdots \to A^{(n)} \simeq k$ in CAlg

descending induction on n: **Claim** $A^{(i)} = C^*(\mathfrak{h}^{(i)})$ for some $\mathfrak{h}^{(i)} \in \mathrm{Lie}^{goo}$

Claim $\overset{2.}{\Rightarrow}$ $A^{(i)} \xrightarrow{\simeq} C^*\mathfrak{D}A^{(i)}$, $\mathfrak{D}(A^{(i)}) \simeq \mathfrak{D}(C^*(\mathfrak{h}^{(i)})) \underset{(a)}{\simeq} \mathfrak{h}^{(i)} \in \mathrm{Lie}^{goo}$

Pf of claim: $i = n$: $A^{(n)} \simeq k = C^*(0)$ $1 \Rightarrow$ ✓

$\mathfrak{h}^{(n)} = 0 \in \mathrm{Lie}^{good}$

$i < n$.

$$\begin{tikzcd}
A^{(i)} \arrow[r] \arrow[d, "\sigma:"'] \arrow[dr, phantom, "\lrcorner", very near start] & k \\
A^{(i+1)} \arrow[r] & k \oplus k[n_i] \arrow[from=1-2]
\end{tikzcd}$$

$\in C^*|_{\mathrm{Lie}^{good}}$ b/c b)

ind. hyp. $C^*|_{\mathrm{Lie}^{good}}$

$\rightsquigarrow$ define X: homotopy po

$$\begin{tikzcd}
\mathfrak{D}(k \oplus k[n_i]) \arrow[r] \arrow[d, "\tau:"'] & \mathfrak{D}(k) \simeq 0 \arrow[d] \\
\mathfrak{D}(A^{(i+1)}) \arrow[r] & \mathfrak{h}^{(i+1)} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

in Lie^{dg}

$\boxed{(c) \Rightarrow \mathfrak{h}^{(i+1)} \in \mathrm{Lie}^{good}}$

~~$v: A \longrightarrow C^*\mathcal{D}(A)$~~ $\quad v: \mathrm{id}_{(\mathrm{CAlg}^{\mathrm{aug}})^{op}} \longrightarrow C^*\mathcal{D}$

induces a map of diagrams

$$\sigma \longrightarrow C^*(\tau)$$

$$\begin{tikzcd}
 & C^*(\overset{C^*(g^{(n_i)})\ =}{\mathcal{D}(k\oplus k[n_i])}) & & \overset{\mathcal{D}\ =\ C^*(0)}{C^*(0)} \arrow[ll] \\
k\oplus k[n_i] \arrow[ur, "2.)\ \simeq"] & & k \arrow[ll] \arrow[ur, "\simeq\ 2.)"'] & \\
 & C^*(\mathcal{D}(\underbrace{A^{(i+1)}}_{=C^*(g^{(i+1)})})) \arrow[uu] & & C^*(g^{(i+1)}) \arrow[ll] \arrow[uu] \arrow[ul, phantom, "\ulcorner"] \\
A^{(i+1)} \arrow[uu, "2.)\ \simeq"'] \arrow[ur] & & k\oplus A^{(i)} \arrow[ll] \arrow[uu] \arrow[ur, "\text{is equivalence too}"'] \arrow[ul, phantom, "\ulcorner"] &
\end{tikzcd}$$

$$\Rightarrow A^{(i)} \simeq C^*(g^{(i+1)})$$

4.) Reduce to $\phi: k \to k\oplus k[n]$ by old stuff.

Then proceed similarly as in 3.) ▨

prorepresentable formal moduli problems

Def'n: $\mathrm{Pro}(\mathrm{CAlg}_k^{\mathrm{sm}}) \subseteq \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})^{\mathrm{op}}$

- containing corepresentable functors $\mathrm{Spf}\, A$ for $A \in \mathrm{CAlg}_k^{\mathrm{sm}}$

$$\mathrm{Spf}\,A : \begin{cases} \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S} \\ B \mapsto \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{sm}}}(A, B) \end{cases}$$

$\mathrm{CAlg}_k^{\mathrm{sm}} \subseteq \mathrm{CAlg}_k^{\mathrm{aug}}$

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{sm}}}(A,B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{\mathrm{CAlg}}(A,B) \arrow[d] \\ \{*\} \arrow[r, "\text{augm.}"'] & \mathrm{Map}_{\mathrm{CAlg}}(A,k) \end{tikzcd}$$

- closed under <u>filtered colimits</u>
  "pro"

Def'n: $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ is <u>pro-representable</u> if it belongs to $\mathrm{Pro}(\mathrm{CAlg}_k^{\mathrm{sm}})$

Lemma: X pro-repr $\Rightarrow$ X is FMP.

Every FMP has a "smooth" presentation ("resolution") by prorepresentable ones:

Prop: $X \in \mathrm{FMP}$. There is a simplicial object $X_\bullet \in \mathrm{FMP}/X$ s.t.

(1) Each X_n is prorepresentable

~~(2) $X_n \to M_n(X_\bullet)$ is smooth "matching object"~~

(2) X is equivalent to the geometric realization $|X_\bullet|$ in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$

Application of "small object argument" for ∞-categories

Remark: Actually, this proposition says more: existence of a "smooth hypercover" of derived alg geom (formal) derived stacks

Missing:

$$\begin{tikzcd} \mathrm{Lie}_k \arrow[r, "\Psi"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathrm{FMP} \arrow[l, bend right, "\Phi \text{ left adj}"'] \end{tikzcd}$$

$u : \mathrm{id}_{\mathrm{FMP}} \longrightarrow \Psi \circ \Phi$ is an equivalence.

$\forall X$ FMP, Want to show: $X \to (\Psi \circ \Phi)(X)$ is equivalence

Recall: Thm: $u : X \to Y$ map of FMP.

u induces equiv. of tangent cplxes
iff
u is equiv. of FMP.

$$X \circ \mathrm{Dual} = \mathbb{T}_X$$
$$\downarrow$$
$$Y \circ \mathrm{Dual} = \mathbb{T}_Y$$

$\mathrm{Dual} = S_*^{fin} \to \mathrm{CAlg}_k$
$\{k \oplus k[n]\}$

So, need to show: $E := \mathrm{Dual} : S_*^{fin} \to \mathrm{CAlg}_k$ from $\{k \oplus k[n]\}$

~~$X \circ$ De~~

$$\theta = u \circ E : X \circ E \longrightarrow \underline{\underline{(\Psi \circ}}\ \Phi)(X) \circ \underline{\underline{E}} \ \simeq\ e(\Phi(X))$$

Recall: $\Psi : \mathrm{Lie}_k \longrightarrow$ Moduli from $\mathrm{CAlg}_k \xrightarrow{\mathfrak{D}} \mathrm{Lie}_k^{op} \xrightarrow{y(\mathfrak{g})} \mathcal{S}$
$\quad\ \ \mathfrak{g} \qquad \Psi(\mathfrak{g}_*) \circ E : S_*^{fin} \xrightarrow{} \mathcal{S}$ = spectrum

$\rightsquigarrow e : \mathrm{Lie}_k \longrightarrow \mathrm{Sp}(\mathcal{S})$

$\theta : X \circ E \longrightarrow e(\Phi(X))$ is equivalence:

Choose simplicial "resolution" $X_\bullet \in$ ~~Moduli~~ $\mathrm{FMP}/_X$ --- X_n pro-repr. etc

~~($A \in \mathrm{CAlg}_k^{sm}$, $X_\bullet(A)$ is hypercovering of $X(A)$~~
~~$\Rightarrow |X_\bullet(A)| \to X(A)$ is homotopy equiv.)~~

$\Big(\Longrightarrow$ X is colimit of $X_\bullet$ in $\mathrm{Fun}(\mathrm{CAlg}_k^{sm}, \mathcal{S})$
in FMP. $\Big)$

$$X \circ E \simeq |X_\bullet \circ E| \quad \text{in } \mathrm{Sp}(\mathcal{S})$$

- Φ left adjoint $\Rightarrow$ comm. w/ small colims
- e preserves sifted colims (FACT.)

$\Big\}$ so in part geom. realiz.

$$\Longrightarrow \quad e(\Phi(X)) \simeq e(\Phi(|X_\bullet|)) \simeq \big| e\Phi(X_\bullet) \big|$$

$\Rightarrow$ θ is geom. realiz. of

$$\theta_\bullet : X_\bullet \circ E \longrightarrow e(\Phi(X_\bullet))$$

Need to show θ_n is equivalence

$$\Updownarrow$$

$$X_n \longrightarrow (\Psi \circ \Phi)(X_n) \quad \text{is equiv.}$$

So, check (ii) for X prorepresentable.

Recall/Learn: Def'n filtered colim of representables

Φ, Ψ commute w/ filtered colims

$\Rightarrow$ enough for $X = ~~\mathrm{Spec}(A)~~$ $A \in \mathrm{CAlg}_k^{sm}$
$X = \mathrm{Spf}\, A$

$$\begin{tikzcd} \mathrm{Lie} \arrow[r, "\Psi"'] \arrow[d, shift right, "C^\vee_{\mathrm{Lie}}"'] \arrow[d, phantom, "\vdash"] & \mathrm{FMP} \arrow[l, bend right, "\Phi"'] \\ (\mathrm{CAlg}_k^{aug})^{op} \arrow[u, shift right, "\mathfrak{D}"'] \arrow[ur, "\mathrm{Spf}"'] & \end{tikzcd}$$

$$\Phi(\mathrm{Spf}\, A) = \mathfrak{D}(A)$$

remains: $\mathrm{Spf}\, A \longrightarrow \Psi(\mathfrak{D}(A))$ is equiv.

i.e. $\forall B \in \mathrm{CAlg}_k^{sm}$,

$$(\mathrm{Spf}\, A)(B) = \mathrm{Map}_{\mathrm{CAlg}_k}(A,B) \xrightarrow{\simeq} \underset{\underset{(\Psi(\mathfrak{D}(A)))(B)}{\shortparallel}}{\mathrm{Map}_{\mathrm{Lie}}(\mathfrak{D}(B), \mathfrak{D}(A))} \simeq \mathrm{Map}_{\mathrm{CAlg}}(A, C^\vee\mathfrak{D}(B))$$

By Thm $B \xrightarrow{\simeq} C^\vee \mathfrak{D}(B)$ ∎