

Use: $B \in (\text{Cat}_{\text{Lie}}^{\text{an}}, \text{an} \cdot \Rightarrow B \rightarrow C^r \mathcal{D}(B)$
 is equiv.

Def'n good Lie algebra \mathfrak{g} : • cofibrant, $\exists V_n \subseteq \mathfrak{g}_n$ st

1. $\forall n \in \mathbb{Z}, V_n$ is f.dim'l

2. $V_n = 0 \quad n \geq 0$

3. \mathfrak{g}_n is a free Lie algebra

Thm $\text{Lie}_k^{\text{good}} \subseteq \text{Lie}_k$ full of cat spanned by good Lie algebras:

(a) $\forall \mathfrak{g} \in \text{Lie}^{\text{good}}, \mathfrak{g} \rightarrow \mathcal{D}C^r \mathfrak{g}$ is equiv

(from Prop. last week)

(b) $\forall n \geq 1 \exists \mathfrak{g}^{(n)} \in \text{Lie}^{\text{good}}$, and an equivalence in $\text{Cat}_{\text{Lie}}^{\text{an}}$
 $k \oplus k[n] \simeq C^r \mathfrak{g}^{(n)}$

$$\mathfrak{g}^{(n)} := \text{Free}_{\text{Lie}}(k[-n-1])$$

(c) $V_n = \mathfrak{g}^{(n)} \simeq \mathcal{D}(k \oplus k[n]) \rightarrow \mathcal{D}(k) \simeq 0$

$$\begin{array}{ccc} \mathfrak{g}^{(n)} & \longrightarrow & 0 \\ \downarrow & \searrow \simeq & \downarrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{g}' \end{array}$$

$$\mathfrak{g} \in \text{Lie}^{\text{good}} \Rightarrow \mathfrak{g}' \in \text{Lie}^{\text{good}}$$

(recall: $B \rightarrow k$
 $\downarrow \quad \searrow \quad \downarrow$
 $A \rightarrow k \oplus k[n]$
 A small \Rightarrow B small

Prop Properties of \mathcal{D} :

1. $\mathcal{D}(k) \cong 0$

2. $A \in \text{CATg}^{\text{aug}}$, $A = C^*(g)$, $g \in \text{Lie}^{\text{good}}$ $\Rightarrow A \rightarrow C^*\mathcal{D}(A)$ is an equiv in CATg^{aug}

3. $A \in \text{CATg}^{\text{aug}}$ is small $\Rightarrow \mathcal{D}(A) \in \text{Lie}^{\text{good}}$, $A \rightarrow C^*\mathcal{D}(A)$ is equiv

4. σ $\begin{matrix} A' \rightarrow B' \\ \downarrow \dashv \quad \downarrow \dashv \\ A \rightarrow B \end{matrix}$ is pb in CATg^{aug} , A, B, ϕ small $\Rightarrow \mathcal{D}(\sigma)$ is pb.

Pf 1. $C^*(0) \cong k$, 0 is good Lie alg $\xrightarrow{\text{Thm (a)}} 0 \cong \mathcal{D}(C^*(0)) = \mathcal{D}(k)$

2. $g \in \text{Lie}^{\text{good}} \xrightarrow{\text{Thm (a)}} g \xrightarrow{\cong} \mathcal{D}C^*(g)$

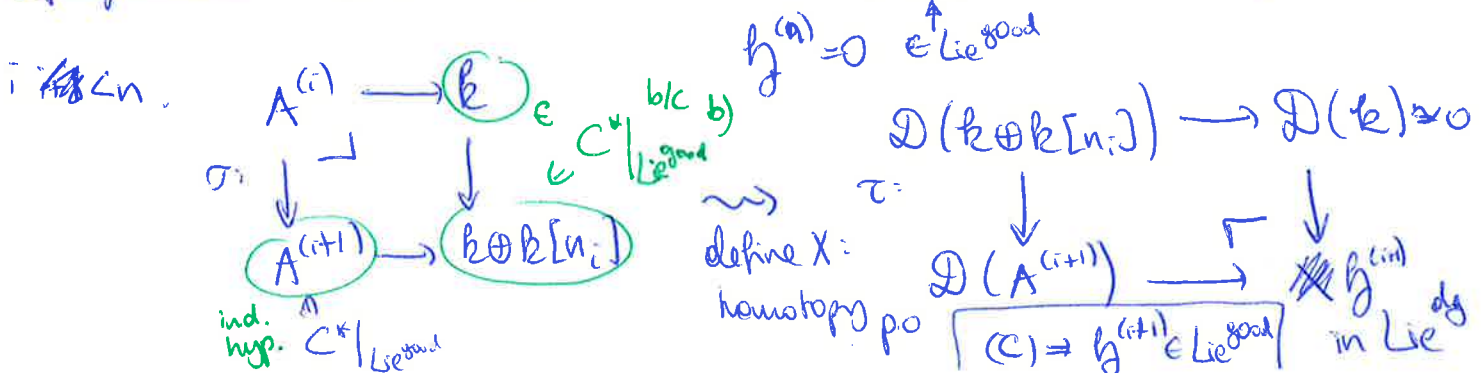
$\Rightarrow A = C^*(g) \xleftarrow{\cong} C^*\mathcal{D}C^*(g) = C^*\mathcal{D}A$

is left homotopy inverse of $u: A \rightarrow C^*\mathcal{D}A$
 b/c u is unit of adj. $\Rightarrow u$ is equivalence as well.

3. A small cda \Rightarrow can choose a sequence of elementary morphisms
 $A = A^{(0)} \rightarrow A^{(1)} \rightarrow \dots \rightarrow A^{(n)} \cong k$ in CATg^{aug}
descending induction on n : Claim $A^{(i)} = C^*(g^{(i)})$ for some $g^{(i)} \in \text{Lie}^{\text{good}}$

Claim $\Rightarrow A^{(i)} \xrightarrow{\cong} C^*\mathcal{D}A^{(i)}$, $\mathcal{D}(A^{(i)}) \cong \mathcal{D}(C^*(g^{(i)}))$

Pf of claim: $i=n$: $A^{(n)} \cong k = C^*(0) \stackrel{1}{\Rightarrow} \checkmark$ $\begin{matrix} \cong \\ \text{(a)} \end{matrix} 0 \in \text{Lie}^{\text{good}}$

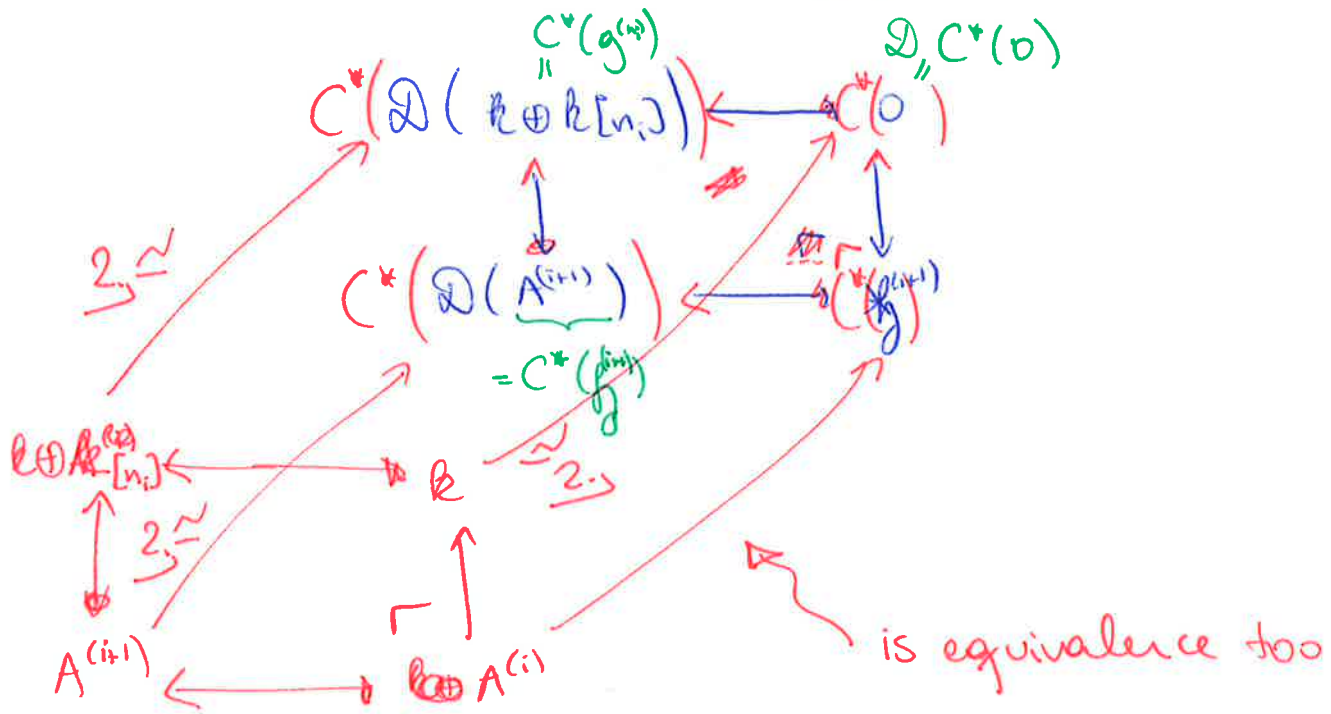


$$u: A \rightarrow C^* \mathcal{D}(A)$$

$$v: \text{id}_{(C^* \mathcal{D}(A))^{op}} \rightarrow C^* \mathcal{D}$$

induces a map of diagrams

$$\sigma \rightarrow C^*(\tau)$$



$$\Rightarrow A^{(i)} \cong C^*(\mathcal{D}(A^{(i)}))$$

4. Reduce to $\phi: k \rightarrow k \oplus k[n]$ by old stuff.
Then proceed similarly as in 3.



prorepresentable formal moduli problems

Def'n: $\text{Pro}(\text{CAlg}_k^{\text{sm}}) \subseteq \text{Fun}(\text{CAlg}_k^{\text{sm}}, \mathcal{S})^{\text{op}}$

) containing corepresentable functors $\text{Spf } A$
for $A \in \text{CAlg}_k^{\text{sm}}$

$$\text{Spf } A: \begin{cases} \text{CAlg}_k^{\text{sm}} \rightarrow \mathcal{S} \\ B \mapsto \text{Map}_{\text{CAlg}_k^{\text{sm}}}(A, B) \end{cases} \begin{array}{c} \longrightarrow \text{Map}_{\text{CAlg}_k^{\text{sm}}}(A, B) \\ \downarrow \\ \{*\} \end{array} \begin{array}{c} \longrightarrow \text{Map}_{\text{CAlg}_k^{\text{sm}}}(A, B) \\ \downarrow \\ \text{Map}_{\text{CAlg}_k^{\text{aug}}}(A, B) \end{array}$$

$$\text{CAlg}_k^{\text{sm}} \subseteq \text{CAlg}_k^{\text{aug}}$$

) closed under filtered colimits
"pro"

Def'n: $X: \text{CAlg}_k^{\text{sm}} \rightarrow \mathcal{S}$ is pro-representable if it belongs to $\text{Pro}(\text{CAlg}_k^{\text{sm}})$

Lemma: X pro-repr $\Rightarrow X$ is FMP.

Every FMP has a "smooth" ^{"resolution"} presentation by prorepresentable ones.

Prop: $X \in \text{FMP}$. There is a simplicial object $X_\bullet \in \text{FMP}/X$ st.

(1) Each X_n is prorepresentable

(1A) $X_n \rightarrow \mathcal{M}_n(X)$ is smooth

(2) X is equivalent to the geometric realization $|X_\bullet|$ in $\text{Fun}(\text{CAlg}_k^{\text{sm}}, \mathcal{S})$
"matching object"

Application of "small object argument" for ∞ -categories

Remark: Actually, this proposition says more:

existence of a "smooth hypercover" of derived alg geom (formal) derived stacks

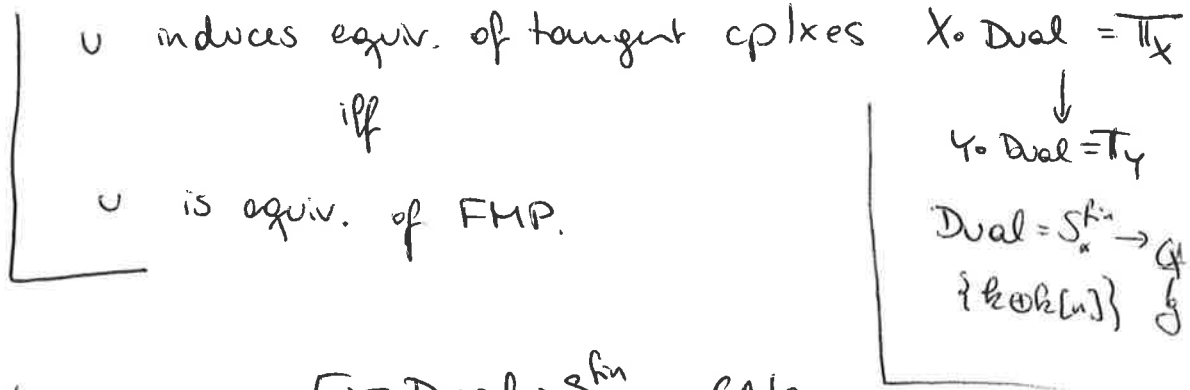
Missing:

$$\text{Lie}_k \xrightleftharpoons[\Psi]{\Phi \text{ left adj}} \text{FMP}$$

$\nu: \text{id}_{\text{FMP}} \longrightarrow \Psi \circ \Phi$ is an equivalence.

$\forall X \text{ FMP}$, want to show: $X \rightarrow (\Psi \circ \Phi)(X)$ is equivalence

Recall: Thm: $\nu: X \rightarrow Y$ map of FMP.



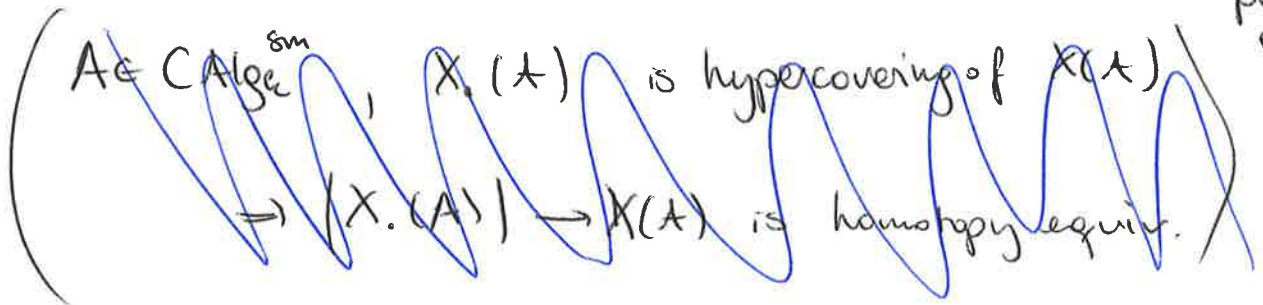
So, need to show: $E := \text{Dual}: S_x^{fin} \rightarrow \text{CAlge}_k$
 from $\{k \otimes k[n]\}$

$$\theta = \nu \circ E: X \circ E \longrightarrow (\Psi \circ \Phi)(X) \circ E \xrightarrow{\sim} e(\Phi(X))$$

Recall: $\Psi: \text{Lie}_k \rightarrow \text{Moduli from } \mathcal{G} \xrightarrow{E} \text{CAlge}_k \xrightarrow{\mathcal{D}} \text{Lie}_k^{op} \xrightarrow{y(\mathfrak{g})} \mathcal{S}$
 $\Psi(\mathfrak{g}_*) \circ E: S_x^{fin} \rightarrow \mathcal{S}$ = spectrum
 $\rightsquigarrow e: \text{Lie}_k \rightarrow \text{Sp}(\mathcal{S})$

$\theta: X \circ E \longrightarrow e(\Phi(X))$ is equivalence:

Choose simplicial "resolution" $X_* \in \text{FMP}/X$ --- X_n pro-repr. etc



$\left(\Rightarrow X \text{ is colimit of } X_i \text{ in } \text{Fun}(\text{CAlg}_{\mathbb{Z}}^{\text{sm}}, \mathcal{S}) \right)$
in FMP.

$$X \circ E \simeq |X \circ E| \text{ in } \text{Sp}(\mathcal{S})$$

- Φ left adjoint \Rightarrow comm. w/ small colims
 - e preserves sifted colims (FACT.)
- } so in part geom. realiz.

$$\Rightarrow e(\Phi(X)) \simeq e(\Phi(|X \circ E|)) \simeq |e\Phi(X)|$$

$\Rightarrow \theta$ is geom. realiz. of

$$\theta : X \circ E \rightarrow e(\Phi(X))$$

Need to show θ_n is equivalence

$$\begin{matrix} \Downarrow \\ X_n \rightarrow (\Psi \circ \Phi)(X_n) \text{ is equiv.} \end{matrix}$$

So, check (iii) for X prorepresentable. Recall/Learn:

Φ, Ψ commute w/ filtered colims

\Rightarrow enough for $X = \text{Spec}(A)$ $A \in \text{CAlg}_{\mathbb{Z}}^{\text{sm}}$
 $X = \text{spf } A$

$$\Phi(\text{spf } A) = \mathcal{D}(A)$$

remains: $\text{spf } A \rightarrow \Psi(\mathcal{D}(A))$ is equiv.

i.e. $\forall B \in \text{CAlg}_{\mathbb{Z}}^{\text{sm}}$,

$$(\text{spf } A)(B) = \text{Map}_{\text{CAlg}_{\mathbb{Z}}}^{\text{sm}}(A, B) \xrightarrow{\cong} \text{Map}_{\text{Lie}}(\mathcal{D}(B), \mathcal{D}(A)) \simeq \text{Map}_{\text{CAlg}_{\mathbb{Z}}}^{\text{sm}}(A, \mathcal{D}(B))$$

By Thm $B \simeq \mathbb{N}^{\mathcal{D}(B)}$

filtered colim of \bullet representables

